Let $z'(\alpha_0) \neq 0$. The condition of positive definiteness of the quadratic form $\delta^2 R$ on the linear manifold $\delta f_3 = 0$:

$$(B \cos\varphi \cos\alpha_0)\, \delta\Omega_1 + (B \sin\varphi \cos\alpha_0)\, \delta\Omega_2 + 2c_3 \sin\alpha_0\, \delta\alpha = 0$$

leads to the single inequality

$$b^2 c_3^4 + 3\,[Pz'(\alpha_0)]^2 + bc_3^2 Pz''(\alpha_0) > 0$$

Thus, for sufficiently large values of the angular velocity, the manifold Σ_3 of the stationary motions of the body is conditionally stable relative to these deviations.

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SHOCK INTERACTION BETWEEN A CONCENTRATED OBJECT AND A ONE-DIMENSIONAL ELASTIC SYSTEM*

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A physical interpretation of the results obtained earlier (**Malanov S.B. and Utkin G.A. Formulation of a problem of shock interaction between a concentrated object and a one-dimensional elastic system. Gor'kii, 1986. Dep. at VINITI 5.12.86, 8304-B86.) for the shock interaction of a homogeneous elastic system with a concentrated object is given in the form of the laws of variation of the energy and moments. The impact of a material point against a string is considered as an example, and the dependence of the time of contact and the coefficient of restitution on the parameters of the problem is given.

The problem of the correct conditions at the point of contact and of relations holding at the beginning and end of contact were solved in /1, 2/, where additional geometrical and physical concepts (laws of conservation of energy, momentum, etc.) were brought in. The study of a

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coordinated continuous interaction based on Hamilton's principle* (*Vesnitskii A.I., Krysov S.V. and Utkin G.A., Formulating the boundary value problems of the dynamics of elastic systems based on the variational Hamilton principle. Ucheb. posobiye, Gor'kii, Gor'k. Un-t, 1983. Vesnitskii A.I., Kaplan L.E., Krysov S.V. and Utkin G.A., Selfcoordinated problems of the dynamics of one-dimensional systems with moving loads and clamps. Preprint 159, Gor'kii, Nauchn. - Issled. Radiofiz. In-t, Akad. Nauk SSSR, 1982.) made it possible to solve completely the problem of the correct conditions at the point of contact without bringing in additional considerations. The use of the same approach in the problem with a finite time of contact, gave relations which were valid at the initial and final instant of the contact.

1. Let us consider a mechanical system consisting of an elastic directrix (one-dimensional system) and a concentrated object. The concentrated object may move along the directrix over a certain period (the time of contact). Thus we arrive at the problem of describing their coordinate motion and of determining the initial and final instant of the period of contact.

Let x be the coordinate along the one-dimensional system, t is the time, $D = \{(x, t)$: $a \leqslant x \leqslant b$, $t_1 \leqslant t \leqslant t_4\}$ is a rectangular region in the plane xt, t_2 and $t_3\ (t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4)$ is the initial and final instant of the contact respectively. We assume that the law of motion of the load is described by some generalized coordinate $z(t)$ and vector functions $\mathbf{u}_0(t), \mathbf{v}_0(t)$. The function z and all components of the vector functions $\mathbf{u}_0$ and $\mathbf{v}_0$ are continuous on $[t_1, t_4]$ and twice continuously differentiable on $(t_k, t_{k+1}), k=1, 2, 3$. The curve $x = z(t), t \in [t_2, t_3]$ and the straight lines $t = t_2$ and $t = t_3$ divide the region D into two parts D_{i}, $i = 1, \ldots, 4$. The law of motion of the distributed system is described by a set of vector functions of generalized coordinates $\mathbf{u}(x, t), \mathbf{w}(x, t)$, continuous in D, and twice continuously differentiable in D_i, $i = 1, \ldots, 4$. In addition, the derivatives of the functions $\mathbf{u}(x, t)$ and $\mathbf{w}(x, t)$ may become discontinuous due to the difference between the velocities of the load and of the one-dimensional object at the initial instant of contact.

Let $L(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_t, \mathbf{w}, \mathbf{w}_x, \mathbf{w}_t)$ be the density of the Lagrange function of the distributed system, and let ${}^0L(t, z, z^\cdot, \mathbf{u}_0, \mathbf{u}_0^\cdot, \mathbf{v}_0, \mathbf{v}_0^\cdot)$ be the Lagrange function of the load where L, ${}^0L$ are twice continuously differentiable functions of their arguments. Moreover, at the time of contact the relation $\mathbf{u}_0(t) = \mathbf{u}(z(t), t)$ holds.

Then the relations minimizing the action functional can be written, in accordance with the Hamilton's principle, in the form

$$L_{\mathbf{u}} - \frac{\partial}{\partial t} L_{\mathbf{u}_t} - \frac{\partial}{\partial x} L_{\mathbf{u}_x} = 0 \tag{1.1}$$

$$L_{\mathbf{w}} - \frac{\partial}{\partial t} L_{\mathbf{w}_t} - \frac{\partial}{\partial x} L_{\mathbf{w}_x} = 0; \quad (x, t) \in D_i, \quad i = 1, \ldots, 4$$

$$[\mathbf{u}(x, t)] = 0, \ [\mathbf{w}(x, t)] = 0 \tag{1.2}$$

$${}^0L_z - \frac{d}{dt}\,{}^0L_{z^\cdot} = [L - (\mathbf{u}_x, L_{\mathbf{u}_x} - z^\cdot L_{\mathbf{u}_t}) - (\mathbf{w}_x, L_{\mathbf{w}_x} - z^\cdot L_{\mathbf{w}_t})] \tag{1.3}$$

$${}^0L_{\mathbf{u}_0} = \frac{d}{dt}\,{}^0L_{\mathbf{u}_0^\cdot} = [L_{\mathbf{u}_x} - z^\cdot L_{\mathbf{u}_t}]$$

$$[L_{\mathbf{w}_x} - z^\cdot L_{\mathbf{w}_t}] = 0, \quad {}^0L_{\mathbf{v}_0} - \frac{d}{dt}\,{}^0L_{\mathbf{v}_0^\cdot} = 0$$

$${}^0L_z - \frac{d}{dt}\,{}^0L_{z^\cdot} = 0, \quad {}^0L_{\mathbf{u}_0} - \frac{d}{dt}\,{}^0L_{\mathbf{u}_0^\cdot} = 0 \tag{1.4}$$

$${}^0L_{\mathbf{v}_0} - \frac{d}{dt}\,{}^0L_{\mathbf{v}_0^\cdot} = 0, \quad t \in [t_1, t_2) \cup (t_3, t_4]$$

$$\{\langle L - (L_{\mathbf{u}_t}, \mathbf{u}_t) - (L_{\mathbf{w}_t}, \mathbf{w}_t)\rangle + {}^0L - {}^0L_{z^\cdot} z^\cdot - ({}^0L_{\mathbf{u}_0^\cdot}, \mathbf{u}_0^\cdot) - ({}^0L_{\mathbf{v}_0^\cdot}, \mathbf{v}_0^\cdot)\} = 0 \tag{1.5}$$

$$\{L_{\mathbf{u}_t}\} = 0, \quad \{L_{\mathbf{w}_t}\} = 0, \quad \{(L_{\mathbf{u}_t}, \mathbf{u}_x) + (L_{\mathbf{w}_t}, \mathbf{w}_x)\} = 0$$

$$\{{}^0L_{\mathbf{u}_0^\cdot}\} = 0, \quad \{{}^0L_{\mathbf{v}_0^\cdot}\} = 0, \quad \{{}^0L_{z^\cdot}\} = 0$$

where

$$[A(x, t)] = A(z(t) + 0, t) - A(z(t) - 0, t), \ t \in [t_2, t_3]$$

$$\{A(x, t)\} = A(x, t_k + 0) - A(x, t_k - 0)$$

$$x \in [a, z(t_k)) \cup (z(t_k), b], \ k = 2, 3$$

$$\langle A(x, t)\rangle = \int_a^b A(x, t)\, dx$$

It is clear that relations (1.1) specify the differential relations for a distributed system, while expressions (1.2) and (1.3) represent the conditions of coupling when the motion is coordinated. Relations (1.4) give the differential equations of motion of the load before and after the contact, and (1.5) specify the coupling conditions at the initial and final instant of the contact.

2. Let us denote by /3/ $\mathbf{p}(x, t)$ the $n+m+l$-dimensional vector whose components represent the density of the generalized momentum corresponding to the generalized coordinates $\mathbf{u}(x, t)$ and $\mathbf{w}(x, t)$ of the distributed system. We shall write this vector, the density of the external force and the internal potential force at the cross-section x in the form

$$\mathbf{p}(x,t) = (L_{u_t^1}, \ldots, L_{u_t^n}, L_{w_t^1}, \ldots, L_{w_t^m}, 0, \ldots, 0)$$
$$\mathbf{Q}(x,t) = (L_{u^1}, \ldots, L_{u^n}, L_{w^1}, \ldots, L_{w^m}, 0, \ldots, 0)$$
$$\mathbf{T}(x,t) = (L_{u_x^1}, \ldots, L_{u_x^n}, L_{w_x^1}, \ldots, L_{w_x^m}, 0, \ldots, 0)$$

Using expressions (1.1), we can confirm that the following relations hold:

$$\frac{\partial \mathbf{p}}{\partial t} + \frac{\partial \mathbf{T}}{\partial x} = \mathbf{Q}, \quad \frac{\partial p^*}{\partial t} + \frac{\partial T^*}{\partial x} = Q^*, \quad \frac{\partial h}{\partial t} + \frac{\partial S}{\partial x} = N$$
$$p^*(x, t) = -(L_{\mathbf{u}_t}, \mathbf{u}_x) - (L_{\mathbf{w}_t}, \mathbf{w}_x)$$

$$T^*(x, t) = L - (L_{\mathbf{u}_x}, \mathbf{u}_x) - (L_{\mathbf{w}_x}, \mathbf{w}_x)$$
$$h(x, t) = -L + (L_{\mathbf{u}_t}, \mathbf{u}_t) + (L_{\mathbf{w}_t}, \mathbf{w}_t)$$
$$S(x, t) = (L_{\mathbf{u}_x}, \mathbf{u}_t) + (L_{\mathbf{w}_x}, \mathbf{w}_t)$$

where p^* is the density of the wave momentum, T^* is the wave pressure flux, $Q^*(x, t) = L_x$ is the wave pressure force density governed by the distributed reflection, h is the density of generalized energy, S is the wave energy flux and $N(x, t) = -L_t$ is the source strength density affecting the parameters of the distributed system.

Similarly, we can represent the total generalized momentum of the load in the form of a $n+m+l$-dimensional vector

$$\mathbf{p}_0(t) = ({}^0L_{u_0^{\cdot 1}}, \ldots, {}^0L_{u_0^{\cdot n}}, 0, \ldots, 0, {}^0L_{v_0^{\cdot 1}}, \ldots, {}^0L_{v_0^{\cdot l}})$$

and the generalized potential force by

$$\mathbf{Q}_0(t) = ({}^0L_{u_0^1}, \ldots, {}^0L_{u_0^n}, 0, \ldots, 0, {}^0L_{v_0^1}, \ldots, {}^0L_{v_0^l})$$

Taking this into account, we can obtain, from relations (1.3) and (1.4), the differential laws governing the change of momentum and load energy during the contact

$$\frac{d\mathbf{p}_0}{dt} = [\mathbf{T} - z^{\cdot}\mathbf{p}] + \mathbf{Q}_0, \quad \frac{dp_0^*}{dt} = [T^* - z^{\cdot}p^*] + Q_0^*$$
$$dh_0/dt = [S - z^{\cdot}h] + N_0$$

and outside the time of contact

$$\frac{d\mathbf{p}_0}{dt} = \mathbf{Q}_0, \quad \frac{dp_0^*}{dt} = Q_0^*, \quad \frac{dh_0}{dt} = N_0$$

Here $p_t^*(t) = {}^0L_{z^{\cdot}}$ is the load momentum, $Q_0^*(t) = {}^0L_z$ is the potential force, $h_0(t) = -{}^0L_0 + {}^0L_{z^{\cdot}}z^{\cdot} + ({}^0L_{\mathbf{u}_0^{\cdot}}, \mathbf{u}_0^{\cdot}) + ({}^0L_{\mathbf{v}_0^{\cdot}}, \mathbf{v}_0^{\cdot})$ is the total energy of the load and $N_0(t) = -{}^0L_t$ is the strength of the forces affecting the load parameters. At the initial and final instant of the contact the momenta and the energy of the system are preserved, and this follows from relations (1.5), which can be written in the form

$$\{\mathbf{p}\} = \{\mathbf{p}_0\} = 0, \{p^*\} = \{p_0^*\} = \{\langle h\rangle + h_0\} = 0$$

The following global laws of variation of momentum and energy hold for the system in toto:

$$\frac{d\mathbf{P}}{dt} = -\Delta\mathbf{T} + \langle\mathbf{Q}\rangle + \mathbf{Q}_0, \quad \frac{dP^*}{dt} = -\Delta T^* + \langle Q^*\rangle + Q_0^*$$
$$dH/dt = -\Delta S + \langle N\rangle + N_0, \quad \Delta A = A(a) - A(b)$$
$$\mathbf{P}(t) = \langle\mathbf{p}\rangle + \mathbf{p}_0, \quad P^*(t) = \langle p^*\rangle + p_0^*, \quad H(t) = \langle h\rangle + h_0$$

where $\mathbf{P}$ is the vector of total generalized momenta of the system, P^* is the total momentum of the wave, and H is the total energy of the system.

3. Let us consider a central impact of a material point on a bounded string at rest. Assuming that the transverse oscillations of the string are small, we shall write the density of its Lagrange's function in the form $L = {}^1/_2 (\rho u_t^2 - N u_x^2)$ where ρ is the running density, N is the tension and $u(x, t)$ is the transverse deflection of the string. We shall specify Lagrange's function for a point of mass m in the form ${}^*L = {}^1/_2 m (z^{\cdot 2} + u_0^{\cdot 2})$. Here $z^{\cdot}$ and $u_0^{\cdot}$ are the velocity components of the point along the x and u axes respectively.

By virtue of the symmetry of the problem, the point will not move along the string, i.e. $z^{\cdot}(t) \equiv 0$. We can write $z(t) \equiv 0$, $u_0(0) = 0$ without loss of generality. Using the first relation of (1.1), the first relation of (1.2) and the second relation of (1.3), we arrive at equations describing the motion of the system at the time of contact

$$\rho u_{tt} - N u_{xx} = 0 \tag{3.1}$$
$$u(-0, t) = u(+0, t) = u_0(t), \quad m u_0^{\cdot\cdot} = [N u_x]$$

Let us add to these equations the boundary and initial conditions

$$u(-a, t) = 0, \quad u(a, t) = 0, \quad a > 0 \tag{3.2}$$
$$u_0(0) = 0, \quad u_0^{\cdot}(0) = -V$$
$$u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 0, & x \neq 0 \\ -V, & x = 0 \end{cases}$$

Using the integral Laplace transformation with respect to time /4/, we obtain the law of motion of the load

$$u_0^{\cdot}(t) = -V\left(e^{-\alpha\tau} + \sum_{n=1}^{\infty} e^{-q}(L_n(2q) - L_{n-1}(2q))\theta(q)\right) \tag{3.3}$$
$$\tau = \frac{ct}{a}, \quad c = \sqrt{\frac{N}{\rho}}, \quad \alpha = \frac{2\rho a}{m}, \quad q = \alpha\tau - 2n\alpha$$
$$L_n(y) = \frac{e^y}{n!}\frac{d^n}{dy^n}(y^n e^{-y})$$

Here τ is dimensionless time, c is the velocity of propagation of the wave along the string, α is a dimensionless parameter characterizing the ratio of the masses of the string and the point, L_n are the Laguerre polynomials and θ is the Heaviside point function.

Let us consider the problem of the time t_3 of the end of contact. The function $u(x, t)$ is continuous everywhere; we therefore have the relation $\{u(x, t_3)\} = 0$, $x \in [-a, a]$ which, in turn, yields the following relations:

$$\{u_x(z(t_3) - 0, t_3)\} = 0, \quad \{u_x(z(t_3) + 0, t_3)\} = 0$$

After the contact has ceased, the function $u(x, t)$ will be twice continuously differentiable (except at the breaks on the characteristics), i.e.

$$u_x(z(t_3) + 0, t_3 + 0) - u_x(z(t_3) - 0, t_3 + 0) = 0$$

This leads to the following condition: $[u_x(x, t_3 - 0)] = 0$. Turning now to the third relation of (3.1), we obtain the following expression for determining the instant of termination of the contact

$$u_0^{\cdot\cdot}(t_3) = 0 \tag{3.4}$$

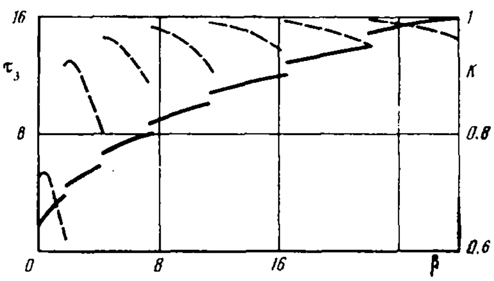
From relations (3.3) and (3.4) it follows that the dimensionless time of termination of contact τ_3 is the first positive root of the algebraic equation

$$1 + \sum_{n=1}^{\infty} F_n(\alpha, \tau)\theta(\tau - 2n) = 0$$
$$F_n(\alpha, \tau) = e^{2n\alpha}(L_n(2\alpha\tau - 4\alpha n) + L_{n-1}(2\alpha\tau - 4\alpha n))$$

The solid lines in the figure show the relation $\tau_3(\beta)$, $\beta \in [0, 28]$ where $\beta = 1/\alpha$, obtained with help of a digital computer. The discontinuous form of this relation results from the discontinuity in the initial velocity of the string (3.2), which in turn was caused by the impact of the point against the string. The discontinuity in $\tau_3(\beta)$ occurs at the "critical" values of β_n, given by the equation

$$1 + \sum_{k=1}^{n} F_k(\alpha_n, 2n + 2) = 0, \quad \beta_n = \frac{1}{\alpha_n}$$

and the magnitude of the discontinuity is found from the equation $\Delta\tau_n = \tau_n - 2n - 2$ where τ_n is the root of the equation



$$1 + \sum_{k=1}^{n+1} F_k(\alpha_n, \tau_n) = 0$$

The dashed lines depict the relation $K(\beta)$ where $K = u_0^{\cdot}(t_3)/V$ is the so-called coefficient of restitution. When $t > t_3$, the separate motion of the objects is found by solving the equations

$$\rho u_{tt} - N u_{xx} = 0, \; u_0^{\cdot\cdot}(t) = 0$$

taking the relations $\{u_t\} = 0$, $\{u_t u_x\} = 0$, $\{u_0^{\cdot}\} = 0$ into account.

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